THE CONFORMAL UNIVERSE I: Physical and Mathematical Basis of Conformal General Relativity

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Abstract

This is the first of a series of three papers on the Principle of Conformal General Relativity (CGR), which generalizes Einstein's Principle of General Relativity (GR) by requiring action–integral invariance under local scale transformations in addition to general coordinate transformations. The idea of a conformal extension of GR was first advanced by Weyl in 1919 and fully developed by Cartan in the early 1920s. For several decades it had little impact on physics, since CGR implies that all fields have zero mass. The interest in this subject revived in the late 1960s, after the discovery that the mass of a field may result from the spontaneous breakdown of a symmetry. In this paper, the main consequences of the spontaneous breakdown of local conformal symmetry are investigated and a number of important consequences are reported. In particular: 1) CGR is possible only in 4D–spacetime; 2) CGR is equivalent to a sort of conformal–invariant GR equipped with a ghost scalar field, here called the *dilation field*; 3) interactions of this field with physical scalar fields result in the production of Higgs fields; 4) CGR satisfies Mach–Einstein's principle; 5) the spontaneous breakdown of conformal symmetry has the properties of a cosmological inflation process which promotes the mutual causation of spacetime–scale expansion and generation of matter.

Contents

1 Introduction

It is not always true that progress in theoretical physics thrives on crisis [39]. Thirty years of string and superstring theory simply have not brought really important results to experimental and observational physics. This is reminiscent of the failure of S–matrix bootstrap paradigm during the 1960s, with the difference that, at that time, there was perhaps an excess of pessimism regarding the predictive power of quantum field theory, whereas in the case of string theory there may have been an excess of optimism regarding the predictive power of algebraic topology. Thus, now we are back at the starting point.

Attempts have been repeatedly made to break through the hard opaque shell surrounding the hypertrophic conceptual apparatus of current theoretical physics, but the basic problem of grand unification is always the same: *General Relativity* (GR) does not smoothly join the Standard Model because of two main difficulties: 1) its implementation in quantum field theory is not renormalizable; 2) it is incapable of explaining inflationary cosmology.

Leaving aside for the moment the first difficulty, with the promise of proposing an honorable solution for it, let us focus on the second, to highlight its main aspects.

Astronomical observations corroborated by the heuristic principle of the non–existence of vantage points in the Universe lead to the following statements: on a large scale, and with respect to a privileged set of *comoving observers*, the universe appears to be *homogeneous* (same from point to point), *isotropic* (same view in all directions) and *in uniform expansion*. Observers are called comoving if each of them sees the others as moving with cosmic expansion. Mutual distances among galactic clusters appear to increase as if all pieces of matter accessible to our observation originated from an explosion diverging from a point in the finite or infinite past and were then subjected to forces depending on their relative distances. However, on intergalactic and galactic scales, expansion uniformity is hampered by the anisotropy of the gravitational field, caused by the inhomogeneity of the distribution of matter.

Gravitational attraction, as described by GR, can only account for decelerations but not for the real time–course of the expansion, as accurate astronomical measure-

ments revealed that it is in moderate acceleration. This fact is currently theoretically explained by the presence of a *cosmological constant* in the GR equation, which was the add–on originally introduced but then rejected by Einstein to support the hypothesis of a stationary universe. However, it is uncertain whether this argument suffices to explain the phenomenon completely, since the details of the expansion are known only approximately. Thus, it is still an open question as to whether the intervention of other cosmic forces must be invoked.

In the GR picture, the problem of the beginning appears insoluble. If all the world–lines of galactic clusters originate from a point–like event, huge amounts of matter and negative gravitational energy would have been initially concentrated in the apex of a light–cone. Alternatively, we would hypothesize that the universe originated from a spherical body of enormous mass - one might guess a huge black hole - which would however raise other serious problems.

The source of all these difficulties is evident if we focus on the structure of Einstein's gravitational equation. Let $g_{\mu\nu}, R_{\mu\nu}, R, \Lambda, \Theta^M_{\mu\nu}, M_{rP} = 2.43 \times 10^{18}$ GeV/c^2 and $\kappa = 1/M^2_{rP}$ be respectively: the *metric tensor*, *Ricci tensor*, *Ricci scalar tensor*, *cosmological constant*, *energy–momentum tensor of matter fields*, *reduced Planck mass*, and *gravitational constant*. The gravitational equation of Einstein

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} = \kappa\, \Theta^M_{\mu\nu}$$

is usually understood as the equation which relates spacetime geometry to the distribution and motion of matter field. Note that, if we interpret

$$\Theta^G_{\mu\nu} = -M^2_{rP}\big(R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu}\big)$$

as an energy–momentum tensor with negative–definite energy, the GR equation takes the symmetric form $\Theta_{\mu\nu} = \Theta^G_{\mu\nu} + \Theta^M_{\mu\nu} = 0$, which may be interpreted as the vanishing condition for the total energy–momentum tensor of geometry and matter - as would also be the case with an empty world.

According to these naive considerations, we may speculate that the universe originated from the fundamental state of an empty world perhaps as a consequence of a quantum fluctuation capable of priming the spontaneous breakdown of some

fundamental symmetry, and thence evolved due to an energy–momentum transfer from geometry to matter. But the idea of such a prodigious zero–sum game must be ruled out immediately from standard GR theory since the continuity equations

$$D^\mu \Theta^G_{\mu\nu} = 0\,, \quad D^\mu \Theta^M_{\mu\nu} = 0\,,$$

where D^μ are contravariant spacetime derivatives, hold separately, the former being imposed by the second Bianchi identities [12] and the latter following by consequence.

No matter how fanciful it may seem, this idea has the merit of suggesting that a suitable modification of GR might account for an effective energy–momentum trade–off between geometry and matter.

The purpose of this and next two papers is to prove that this is indeed the case for the conformal extension of GR proposed by Weyl in 1919 [40] and fully developed by Cartan on a purely geometric ground in the early 1920s [5] [7] [8] [6] - hereafter referred to as *Conformal General Relativity* (CGR) - in which a new degree of freedom accounting for a possible scale expansion of spacetime geometry is introduced. I shall formulate the problem for spacetimes of dimension $n > 2$ (briefly, nD spacetimes) precisely to prove that it can be implemented only in 4D.

This idea is not new. Several authors have contributed to it since the early 1960s. Suffice it to mention Gürsey (1963) [19], Schwinger (1969) [34], Mack and Salam (1969) [26], Parker (1969, 1971) [29] [30], Fubini (1976) [15], Grib *et al.* (1976) [16], Englert *et al.* (1976) [14], Coleman and Callan (1977) [9] [10], Brout *et al.* (1978, 1979) [2] [3], Guth (1981, 1993) [17] [18], and many others. Unfortunately, these studies, although conceptually appealing, failed to reach their goal, perhaps because of insufficient knowledge, wrong guesses, and a certain distraction caused by the surge of expectations elicited by string theory.

My interest in this subject is quite recent and mainly motivated by the vivid impression that the excellent ideas and interesting contributions advanced by the above–mentioned authors were ranged like the pieces of a puzzle waiting for solution.

2 Riemann manifolds and Cartan manifolds

Both GR and CGR describe the universe as grounded on a differentiable nD manifold enveloped by a principal bundle formed of isomorphic representations of a finite continuous group, called the *fundamental group*. The fundamental group of GR is the nD Poincaré group and that of CGR is its conformal extension. The structures of both groups can easily be determined according to the transformation properties of a fundamental tensor $g_{\mu\nu}$ of pseudo-Euclidean signature $(+, - \ldots, -)$.

The local representations of the fundamental group are related to each other by a differential law called the *manifold connection*, which describes the variations in fundamental–group representations as detectable from a local inertial frame driven along a path. Connections which do not return the identity when local frames are driven along closed paths are said to possess a *curvature*.

In this general scheme, physical particles are described as irreducible representations of the fundamental group which, in general, undergo path–dependent transformations when the local frame is driven along manifold paths. Local connection curvatures are the group–transformation residues per unit area of connections carried along infinitesimal closed paths, and represent local forces acting on particles which may be grounded on the manifold.

Different groups and types of curvatures characterize different manifolds. Since in GR the fundamental group preserves the metric character of spacetime geometry, the manifold is a *Riemann manifold*. Instead, the fundamental group of CGR is the group of conformal transformations in nD, which is the closest light–cone–preserving extension of the Poincaré group in nD containing the subgroup of local dilations. We refer to this manifold as a *Cartan manifold*.

2.1 Riemann manifolds and local metric invariance

Einstein's Principle of GR asserts the invariance under general coordinate transformations of the action integral grounded on a nD Riemann manifold (with $n > 2$) parameterized by n adimensional coordinates $x^\mu, \mu = 0, 1, \ldots, n-1$, which are presumed to map the entire manifold over a topographic collection of connected

charts. General coordinate transformations of the form $x^\mu \to \bar{x}^\mu(x)$ are presumed to preserve length and volume elements. Hence, denoting by $g_{\mu\nu}(x)$ the fundamental tensor of the manifold and by $ds^2(x) = g_{\mu\nu}(x)\, dx^\mu dx^\nu$ the squared line element, the change in coordinates will produce the transformations

$$g_{\mu\nu}(x) \to \bar{g}_{\mu\nu}(\bar{x}), \quad ds(x) \to d\bar{s}(\bar{x}),$$

so as to satisfy the equalities

$$d\bar{s}^2(\bar{x}) = \bar{g}_{\mu\nu}(\bar{x}) d\bar{x}^\mu d\bar{x}^\nu = g_{\mu\nu}(x) dx^\mu dx^\nu = ds^2(x); \quad \sqrt{-\bar{g}(\bar{x})}\, d^n\bar{x} = \sqrt{-g(x)}\, d^n x.$$

However, the parameterization can be rescaled so as to preserve the unit volume element, posing, for instance, $\sqrt{-g(x)} = 1$ [12]. Riemann connections are characterized by the property that translations have zero curvature, while Lorentz rotations generally have non–zero curvature. This means that any infinitesimal round–trip of the local reference frame from a point x to the same point x generally results in an infinitesimal Lorentz rotation of the frame at x. This rotation, which ultimately accounts for the effects of gravitational forces and/or local–frame accelerations, is fully represented by the local components of the Riemann tensor $R_{\mu\nu\rho\sigma}(x)$.

Denoting the total action integral of the universe as $A = A^G + A^M$, where

$$\begin{aligned} A^G &= -\frac{1}{2\kappa}\int \sqrt{-g}\big[R(g^{\mu\nu}, \partial_\lambda g^{\mu\nu}) - 2\Lambda\big] d^n x\,, \\ A^M &= \int \sqrt{-g}\, L^M(g^{\mu\nu}, \partial_\lambda g^{\mu\nu}, \mathbf{\Psi}, \partial_\lambda \mathbf{\Psi})\, d^n x\,, \end{aligned}$$

are respectively the action integral of the geometry (characterized by negative–definite variations of kinetic energy) and the action integral of matter (characterized by positive–definite variations); R is the Ricci scalar as a function of $g^{\mu\nu}$ and their partial derivatives, Λ the cosmological constant, L^M the Lagrangian density of the matter as a functional of the metric tensor and a set of matter fields $\mathbf{\Psi}$. We can then derive the GR gravitational equation from the variational equation

$$\frac{\delta A}{\delta g^{\mu\nu}(x)} \equiv \frac{\delta(A^G + A^M)}{\delta g^{\mu\nu}(x)} = 0\,,$$

stating the *invariance of the total action under general coordinate transformations.* Since we have

$$\frac{2}{\sqrt{-g}}\frac{\delta A^G}{\delta g^{\mu\nu}} = -\frac{1}{\kappa}\Big(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R - \Lambda g_{\mu\nu}\Big) = \Theta^G_{\mu\nu}\,, \tag{1}$$

$$\frac{2}{\sqrt{-g}}\frac{\delta A^M}{\delta g^{\mu\nu}} = 2\Big(\frac{\partial L^M}{\delta g^{\mu\nu}} - \partial_\lambda\frac{\partial L^M}{\delta(\partial_\lambda g^{\mu\nu})} - \frac{g_{\mu\nu}}{2}L^M\Big) = \Theta^M_{\mu\nu}\,, \tag{2}$$

the gravitational equation takes the form $\Theta_{\mu\nu} = \Theta^G_{\mu\nu} + \Theta^M_{\mu\nu} = 0$ already considered in the Introduction. The match between the signs of the two energy–momentum tensors and those of the action integrals is evident.

2.2 Cartan manifolds and local conformal invariance

nD Cartan manifolds differ from nD Riemann manifolds in that the fundamental group of manifold connections incorporates the subgroup of Weyl transformations. These are defined by their action on the line element ds, as follows

$$ds(x) \to d\tilde{s}(x) = e^{\alpha(x)}ds(x)\,,$$

where $\alpha(x)$ is a differentiable real function of manifold parameters, and on the metric tensor and its determinant, as follows

$$g_{\mu\nu}(x) \to \tilde{g}_{\mu\nu}(x) = e^{2\alpha(x)}g_{\mu\nu}(x)\,; \quad g^{\mu\nu}(x) \to \tilde{g}^{\mu\nu}(x) = e^{-2\alpha(x)}g^{\mu\nu}(x)\,; \tag{3}$$

$$\sqrt{-g(x)}\,d^nx \to \sqrt{-\tilde{g}(x)}\,d^nx = e^{n\alpha(x)}\sqrt{-g(x)}\,d^nx\,. \tag{4}$$

Since equation $ds(x) = d\tilde{s}(x) = 0$ defines the same light–cone on the tangent manifold, all light-cones stemming from the points of the manifold are preserved by Weyl transformations.

In the following, all quantities relating or belonging to the Cartan manifold are marked by a tilde.

The extension of Riemann connections by Weyl transformations implies the extension of the fundamental group of the Riemann connection, i.e., the Poincaré group, to the *conformal group* in nD. The latter includes not only the subgroup of

local dilations, which is trivial, but also the subgroup of *special conformal transformations* - or *elations* (term coined by Cartan in 1922 [5]) - the structure of which is described in Section 1 of Part II.

Since we wish to exclude the possibility that round–trips carried on the manifold can alter the size of a body with respect to the local frame, dilation connections must have zero curvature. It can therefore be proved that elation connections also have zero curvature. So, were it not for the existence of infinite time–like paths, both dilation and elation connections would drive trivial gauge transformations and could therefore be completely removed.

In fact, the prominent property of Cartan manifolds is that scale factor $e^{\alpha(x)}$ can be interpreted as the physical factor of universe expansion. This is possible, provided that the Weyl transformations act on any field representation Ψ according to the law

$$\Psi(x) \to \tilde{\Psi}(x) = e^{w_\Psi \alpha(x)}\Psi(x)\,,$$

where w_Ψ is called the *dimension*, or *weight*, of Ψ. These dimensions are uniquely determined by the condition that the total dimension of each conformal–invariant action term is zero.

The transition from Riemann to Cartan connections essentially deeply alters the theory because the transition from Poincaré–group invariance to conformal–group invariance forces the total Lagrangian density not to contain dimensional constants, implying that all particles have zero mass. Clearly, this is physically acceptable, provided that the conformal symmetry is spontaneously broken. Regarding this point, the following theorem is of particular importance:

A necessary (but not sufficient) condition for the action integral of a Poincaré–invariant Lagrangian density $L(x)$ to be conformal invariant is not only that it is free from dimensional constants - which is quite obvious, since conformal invariance implies global scale invariance - but also that trace $\Theta(x) = g^{\mu\nu}(x)\,\Theta_{\mu\nu}(x)$ of its energy–momentum tensor $\Theta_{\mu\nu}(x)$ vanishes.

Proof: Let $A = \int \sqrt{-g(x)}\, L(x)\, d^n x$ be a Poincaré–invariant action integral. Conformal invariance implies the vanishing of the variation of A under infinitesimal

Weyl transformations of the form

$$g^{\mu\nu}(x) \to g^{\mu\nu}(x) + \delta_\epsilon\, g^{\mu\nu}(x) \equiv g^{\mu\nu}(x) - 2\,\epsilon(x)\, g^{\mu\nu}(x)\,,$$

where $\epsilon(x)$ is an arbitrary infinitesimal function of x. We have, therefore,

$$\delta_\epsilon A = -2\int \sqrt{-g(x)}\,\epsilon(x)\, g^{\mu\nu}(x)\,\Theta_{\mu\nu}(x)\, d^n x = -2\int \sqrt{-g(x)}\,\epsilon(x)\,\Theta(x)\, d^n x = 0\,,$$

from which the equation $\Theta(x) = 0$ follows.

2.3 Basics of metric–tensor and conformal–tensor calculus

For the sake of clarity, and for the purpose of indicating a few sign conventions, we list here the basic formulae of tensor calculus involved in this work:

- *Christoffel symbols and their fundamental–tensor variations:*

  $$\Gamma^\lambda_{\mu\nu} = \frac{1}{2} g^{\rho\lambda}\big(\partial_\mu g_{\rho\nu} + \partial_\nu g_{\rho\mu} - \partial_\rho g_{\mu\nu}\big)\,, \quad \delta\Gamma^\lambda_{\mu\nu} = \frac{1}{2} g^{\rho\lambda}\big(D_\mu \delta g_{\rho\nu} + D_\nu \delta g_{\rho\mu} - D_\rho \delta g_{\mu\nu}\big)\,,$$

  where D_μ are the covariant derivatives and $\delta g_{\mu\nu}(x)$ a small variation of $g_{\mu\nu}(x)$.

- *Riemann tensor and its fundamental–tensor variations:*

  $$\begin{aligned} R^\rho_{\ .\,\mu\sigma\nu} &= \partial_\sigma \Gamma^\rho_{\nu\mu} - \partial_\nu \Gamma^\rho_{\mu\sigma} + \Gamma^\lambda_{\mu\nu}\Gamma^\rho_{\lambda\sigma} - \Gamma^\lambda_{\mu\sigma}\Gamma^\rho_{\lambda\nu}\,, \\ \delta R^\rho_{\ .\,\mu\sigma\nu} &= \frac{1}{2}\, g^{\rho\lambda}\big(D_\rho D_\lambda\, \delta g_{\mu\nu} + D_\nu D_\mu\, \delta g_{\lambda\sigma} + D_\nu D_\sigma\, \delta g_{\lambda\mu} - D_\sigma D_\nu\, \delta g_{\lambda\nu} - \\ &\quad D_\sigma D_\nu\, \delta g_{\lambda\mu} - D_\nu D_\lambda\, \delta g_{\mu\sigma}\big) \end{aligned}$$

- *Ricci tensors and their fundamental–tensor variations:*

  $$\begin{aligned} R_{\mu\nu} &\equiv R^\rho_{\ .\,\mu\rho\nu} = \partial_\rho \Gamma^\rho_{\nu\mu} - \partial_\nu \Gamma^\rho_{\mu\rho} + \Gamma^\lambda_{\mu\nu}\Gamma^\rho_{\lambda\rho} - \Gamma^\lambda_{\mu\rho}\Gamma^\rho_{\lambda\nu}\,, \\ \delta R_{\mu\nu} &= \frac{1}{2}\big(D^\rho D_\mu\, \delta g_{\rho\nu} + D^\rho D_\nu\, \delta g_{\rho\mu} - D^2\, \delta g_{\mu\nu} - D_\mu D_\nu g^{\rho\sigma}\, \delta g_{\rho\sigma}\big)\,, \\ R &\equiv R_{\mu\nu} g^{\mu\nu}\,, \quad \delta R = R_{\mu\nu}\, \delta g^{\mu\nu} + (g_{\mu\nu} D^2 - D_\mu D\nu)\, \delta g^{\mu\nu}\,, \end{aligned} \tag{5}$$

  where D^2 is the covariant D'Alembert operator. The last equation yields the useful formula:

  $$\frac{1}{\sqrt{-g}}\frac{\delta}{\delta g^{\mu\nu}}\int \sqrt{-g}\, f\, R\, d^n x = f\big(R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R\big) + \big(g_{\mu\nu} D^2 - D_\mu D_\nu\big) f\,.$$

The sign convention for the Riemann tensor is that of Eisenhart [12], but that of the Ricci tensors is opposite to Eisenhart's and matches Landau–Lifchitz [23].

The tensor calculus on Cartan manifolds is enriched by new properties, which are completely explained by the Weyl transformations of a few basic quantities. In particular, the following ones are of decisive importance for our investigation. Taking as fundamental–tensor variation the finite transformation $g_{\mu\nu}(x) \to e^{2\alpha(x)} g_{\mu\nu}(x)$, we obtain the Weyl transformations

$$\Gamma^\lambda_{\mu\nu} \to \tilde{\Gamma}^\lambda_{\mu\nu} = \Gamma^\lambda_{\mu\nu} + \delta^\lambda_\nu \partial_\mu \alpha + \delta^\lambda_\mu \partial_\nu \alpha - g_{\mu\nu} \partial^\lambda \alpha\,; \tag{6}$$

$$\begin{aligned} R_{\mu\nu} \to \tilde{R}_{\mu\nu} = R_{\mu\nu} - (n-2)\big[&g_{\mu\nu} g^{\rho\sigma} (\partial_\rho \alpha)(\partial_\sigma \alpha) - (\partial_\mu \alpha)(\partial_\nu \alpha) + \\ &D_\mu \partial_\nu \alpha\big] - g_{\mu\nu} D^2 \alpha\,; \end{aligned} \tag{7}$$

$$R \to \tilde{R} = e^{-2\alpha}\big[R - (n-1)(n-2) g^{\rho\sigma} (\partial_\rho \alpha)(\partial_\sigma \alpha) - 2(n-1) D^2 \alpha\big]\,; \tag{8}$$

where δ^ν_μ is the Kronecker delta and D_μ, D^2 are respectively the covariant differential operators constructed out of $g_{\mu\nu}(x)$ (see, e.g., Ref.[12], with opposite sign conventions for R and $R_{\mu\nu}$). Eqs.(6)–(8) describe the structural changes of the basic tensors of the absolute differential calculus when passing from GR to CGR on an nD manifold.

Another important property of Weyl transformations regards the totally traceless part $C_{\mu\nu\rho\sigma}$ of Riemann tensor $R_{\mu\nu\rho\sigma} = C_{\mu\nu\rho\sigma} + \cdots$, known as the *conformal–curvature tensor of Weyl*, which satisfies the equations

$$C_{\mu\nu\rho\sigma} g^{\mu\nu} = C_{\mu\nu\rho\sigma} g^{\mu\rho} = C_{\mu\nu\rho\sigma} g^{\mu\sigma} = C_{\mu\nu\rho\sigma} g^{\nu\rho} = C_{\mu\nu\rho\sigma} g^{\nu\sigma} = C_{\mu\nu\rho\sigma} g^{\rho\sigma} = 0\,.$$

The property consists precisely of the invariance of mixed tensor $C^{\mu}_{\,.\,\nu\rho\sigma} = g^{\mu\lambda} C_{\lambda\nu\rho\sigma}$ under Weyl transformations, which may then be abbreviated to

$$C^{\mu}_{\,.\,\nu\rho\sigma}(x) \to \tilde{C}^{\mu}_{\,.\,\nu\rho\sigma}(x) = C^{\mu}_{\,.\,\nu\rho\sigma}(x)\,.$$

This means that squared Weyl–tensor $C^2(x) = C_{\mu\nu\rho\sigma}(x)\, C^{\mu\nu\rho\sigma}(x)$ undergoes the Weyl transformation

$$C^2(x) \to \tilde{C}^2(x) = e^{-4\,\alpha(x)} C^2(x).$$

Consequently, the integral

$$\int \sqrt{-g(x)}\, C^2(x)\, d^n x$$

is conformal invariant only for $n = 4$.

3 Conformal–invariant action integrals

On an nD manifold, field dimensions are determined by the condition that the total dimension of conformal–invariant action integrand $\sqrt{-g}\,L$, where L is a Lagrangian density, be zero. In accordance with this rule, scalar fields must have dimension $1 - n/2$ and spinor fields dimension $(1-n)/2$. Since spacetime coordinates x^μ are zero–dimensional parameters, partial derivatives ∂_μ too are zero–dimensional. Since $ds^2 = g_{\mu\nu}\,dx^\mu dx^\nu$ has dimension 2, $g_{\mu\nu}$ has dimension 2, $\sqrt{-g}$ dimension n and Ricci scalar R dimension -2. For consistency with expressions like $\partial_\mu - ieA_\mu$, covariant gauge fields A_μ have dimension zero. However, their contravariant counterparts A^μ have dimension -2 since $g^{\mu\nu}$ has dimension -2 (in virtue of equation $g^{\mu\lambda}g_{\lambda\nu} = \delta^\mu_\nu$).

Here are the main properties of conformal invariant action integrals:

3.1 Action integrals of scalar fields are conformal invariant only in 4D

Proof: In nD, the more general Lagrangian density of a scalar field φ with self–interaction constant c, interacting with the gravitational field through the metric tensor $g_{\mu\nu}(x)$ and its derivatives, and free from dimensional constants [15], has the general form

$$L_\varphi = \frac{1}{2}\Big[g^{\mu\nu}(\partial_\mu\varphi)(\partial_\nu\varphi) + a\,\varphi^2 R - \frac{c(n-2)}{n}\,\varphi^{2n/(n-2)}\Big]\,,$$

where R is the Ricci scalar and a a suitable real constant. The motion equation is

$$D^2\varphi - aR\varphi + c\,\varphi^{(n+2)/(n-2)} = 0\,,$$

where $D^2 = D^\mu D_\mu$ and D_μ are respectively the covariant D'Alembert operator and the covariant derivatives constructed out of $g_{\mu\nu}$. The (improved) energy–momentum tensor [4] is

$$\begin{aligned}\Theta^\varphi_{\mu\nu} \;&=\; (\partial_\mu\varphi)(\partial_\nu\varphi) - \frac{g_{\mu\nu}}{2}\Big[g^{\rho\sigma}(\partial_\rho\varphi)(\partial_\sigma\varphi) - \frac{c(n-2)}{n}\,\varphi^{2n/(n-2)}\Big] + \\ &\quad a\big(g_{\mu\nu}D^2 - D_\mu\partial_\nu\big)\,\varphi^2 + a\,\varphi^2\Big(R_{\mu\nu} - \frac{1}{2}\,g_{\mu\nu}R\Big)\,;\end{aligned}$$

for the computation of which Eq.(5), shown in the previous subsection, was exploited, an integration by parts was performed, and a surface term suppressed.

By contraction with $g^{\mu\nu}$ and the use of the motion equation, the trace of the energy–momentum tensor vanishes only if $a = (n-2)/4(n-1)$, which implies that the action integral can be conformal invariant only if the Lagrangian density has the form

$$L_\varphi = \frac{1}{2}\Big[g^{\mu\nu}(\partial_\mu\varphi)(\partial_\nu\varphi) + \frac{n-2}{4(n-1)}\,R\,\varphi^2 - \frac{c(n-2)}{n}\,\varphi^{2n/(n-2)}\Big]\,.$$

Now, Weyl transformations $\varphi \to \tilde{\varphi} = e^{-\alpha}\varphi$ together with those reported in Eqs. (3), (4) and (8) of subsec.(2.2), produce the transformation

$$\sqrt{-g}\,L_\varphi \to \sqrt{-\tilde{g}}\,\tilde{L}_\varphi =$$
$$\sqrt{-g}\,e^{(n-4)\alpha}\Big\{L_\varphi - \frac{n-4}{8}\Big[n\varphi^2(\partial^\mu\alpha)(\partial_\mu\alpha) - 2(\partial^\mu\varphi^2)(\partial_\mu\alpha)\Big] - \frac{n-2}{4}D^\mu(\varphi^2\partial_\mu\alpha)\Big\},$$

showing that the action integral $\int\sqrt{-g}\,L_\varphi d^n x$ is conformal invariant (up to a surface term) only if $n = 4$. In summary, the only type of conformal–invariant Lagrangian density for a single scalar field φ interacting with the gravitational field is possible only in 4D, and has the general form

$$L_\varphi = \frac{1}{2}\,g^{\mu\nu}(\partial_\mu\varphi)(\partial_\nu\varphi) + \frac{R}{12}\,\varphi^2 - \frac{c}{4}\,\varphi^4\,,$$

from which we obtain motion equation $D^2\varphi - R\varphi/6 + c\,\varphi^3$.

3.2 The action integral of a ghost scalar field has a geometric meaning

At this point, we might think that a gravitational equation similar to that of Einstein comes spontaneously into play, provided that we assume that φ has a non–zero vacuum expectation value (VEV). Unfortunately, however, in this way the gravitational constant would have the wrong sign and consequently gravitational forces would be repulsive.

Instead, let us assume that the action integral has the negative form

$$A^G = -\int\frac{\sqrt{-g}}{2}\Big[g^{\mu\nu}(\partial_\mu\sigma)(\partial_\nu\sigma) + \frac{R}{6}\,\sigma^2 + \frac{\bar{c}}{2}\,\sigma^4\Big]d^4x\,,$$

where $\bar{c}$ is a self–interaction constant, and that $\sigma(x)$ is always positive. This is possible because the motion equation is invariant under the substitution $\sigma \to -\sigma$. Then, without loss of generality, we can pose $\sigma(x) = \sigma_0\,e^{\alpha(x)}$.

This means that $\sigma(x)$ is a scalar field bearing negative kinetic energy, i.e., a ghost field potentially invested with geometric meaning.

It is now evident that, if σ has non–zero VEV, the factor of the Ricci scalar takes the right sign for gravity to be attractive. Even better than this, let us assume

$$\sigma_0 = \sqrt{6/\kappa} = \sqrt{6}M_{rP} \approx 5.96 \times 10^{18}\,\mathrm{GeV/c^2} \approx 4.28 \times 10^{34}\,\mathrm{m}^{-1}(\text{in natural units})$$

and perform the Weyl transformations $g^{\mu\nu} \to \tilde{g}^{\mu\nu} = e^{-2\alpha}g^{\mu\nu}$ and $\sigma(x) \to \tilde{\sigma}(x) = e^{-\alpha}\sigma(x)$. Then the action integral on the Cartan manifold immediately takes the Hilbert–Einstein form

$$\tilde{A}^G = -\int \frac{\sqrt{-\tilde{g}}}{2\kappa}\big(\tilde{R} - 2\Lambda\big)d^4x\,, \quad \Lambda = -3\,\bar{c}\,\sigma_0^2/2 = -18\,\bar{c}\,M_{rP}^2\,, \tag{9}$$

where Λ plays the role of cosmological constant. Since cosmologists estimate it to be of the order of -10^{-52} m^{-2} in natural units, we find $\bar{c} \approx 3.65 \times 10^{-122}$, which is enormously smaller than that predicted by quantum field theory under the assumption that the energy density of the vacuum state is uniquely determined by the zero–point energy of physical fields [39]. Note the negative sign of Λ implies $\bar{c} > 0$, i.e., a *positive potential energy for the ghost field*, and that choosing the positive sign for σ_0 is equivalent to assuming the spontaneous breakdown of conformal symmetry in such a way that the degree of freedom of the ghost field becomes the determinant of the fundamental tensor of the Cartan manifold.

Of note, more general sorts of conformal–invariant geometric actions on the Riemann manifold may include the squared curvature tensor of Weyl $C^2(x)$ in 4D, described in subsec.(2.2), and terms of the dilation field $\sigma(x)$ interacting with some matter fields $\mathbf{\Psi}(x)$, according to the general expression

$$A^G = -\int \frac{\sqrt{-g}}{2}\Big[\frac{\beta^2}{2}\,C^2 + g^{\mu\nu}(\partial_\mu\sigma)(\partial_\nu\sigma) + \frac{R}{6}\,\sigma^2 + 2U(\sigma^2, \mathbf{\Psi})\Big]d^4x\,. \tag{10}$$

Here, β is a real constant and $U(\sigma^2, \mathbf{\Psi})$ is a conformal–invariant positive–definite potential energy term accounting for possible $\sigma \leftrightarrow \mathbf{\Psi}$ interactions. This must be of even degree in σ, otherwise the solution to the motion equation may not always be definite–positive, as required by the exponential form of Weyl transformations.

Note that the Lagrangian density of $\sigma(x)$ must be ascribed to the action integral of the geometry, since its kinetic term bears the negative sign. Also note that an interaction–energy term of the form $U_0(\sigma^2) \propto \sigma^4$ can account for a cosmological constant of the action integral on the Cartan manifold, since the Weyl transformation acts on it as

$$\sqrt{-g}\, U_0(\sigma^2) \propto \sqrt{-g}\, \sigma^4 \to \sqrt{-\tilde{g}}\, \sigma_0^4 \, .$$

A total conformal–invariant action of the system on the Riemann manifold can then be generally written as $A = A^G + A^M$, where

$$A^M = \int \sqrt{-g}\, L_{\mathbf{\Psi}}(g_{\mu\nu}, \partial_\lambda g_{\mu\nu}, \mathbf{\Psi}, D_\mu \mathbf{\Psi})\, d^4x \, , \qquad (11)$$

is the conformal–invariant action–integral over the Riemann manifold of the matter fields interacting with the gravitational field through the metric tensor $g_{\mu\nu}(x)$. It is then evident that all dimensional constants possibly appearing in the spontaneously broken action integral over the Cartan manifold must depend on the VEV of the dilation field, which vanishes, provided that σ_0 vanishes.

Passing from the Riemann to the Cartan manifold by a Weyl transformation, Eq.(10) takes the form

$$\tilde{A}^G = -\int \frac{\sqrt{-\tilde{g}}}{2}\Big[\frac{\beta^2}{2}\,\tilde{C}^2 + \frac{\sigma_0^2}{6}\tilde{R} + 2\tilde{U}(\sigma_0^2, \tilde{\mathbf{\Psi}})\Big] d^4x \, , \qquad (12)$$

which differs from Eq.(9) by the inclusion of the squared Weyl–tensor term and interaction–potential term $\tilde{U}(\sigma_0^2, \tilde{\mathbf{\Psi}})$, which we do not hesitate to split into a part $\tilde{U}_0(\sigma_0^2)$ independent of $\tilde{\mathbf{\Psi}}$ and a part $\tilde{U}^M(\sigma_0^2, \tilde{\mathbf{\Psi}})$, containing only products of σ_0^2 by physical–field monomials. Thus, we can write $\tilde{U}(\sigma_0^2, \tilde{\mathbf{\Psi}}) = -\Lambda + \tilde{U}^M(\sigma_0^2, \tilde{\mathbf{\Psi}})$. It is then evident that $\tilde{U}^M$ may contain mass–terms for one or more physical fields, so that we can shift it from geometric action integral $\tilde{A}^G$ to matter action integral $\tilde{A}^M$.

Performing these substitutions, we can rearrange the action integral on the Cartan manifold as $\tilde{A} = \tilde{A}^{CG} + \tilde{A}^{CM}$, where

$$\tilde{A}^{CG} = -\int \frac{\sqrt{-\tilde{g}(x)}}{2}\Big[\frac{\beta^2}{2}\,\tilde{C}^2 + M_{rP}^2\, \tilde{R} - 2\Lambda\Big] d^4x \, ; \qquad (13)$$

$$\tilde{A}^{CM} = \int \sqrt{-\tilde{g}}\, \tilde{L}^{CM}\, d^4x \equiv \int \sqrt{-\tilde{g}}\, \big[\tilde{L}_{\mathbf{\Psi}} - \tilde{U}^M(\sigma_0^2, \mathbf{\Psi})\big] d^4x \, . \qquad (14)$$

Clearly, in this representation the underlying conformal symmetry of both $\tilde{A}^{CG}$ and $\tilde{A}^{CM}$ appears explicitly broken. So, were not for the presence in the Lagrangian density of $\tilde{A}^{CG}$ of Weyl term $\beta^2\,\tilde{C}^2(x)$, Eq.(13) would appear formally equal to the corresponding expression of the geometric action integral in standard GR.

It is here the case to note that, as we show in Section 6, any small value of $\beta > 0$ suffices to guarantee the renormalizability of gravitational field. Hence, in the semi–classical approximation, which we are working in, we are allowed to simply assume $\beta = 0$ and the geometric action integral as in Eq.(9).

As regards matter–action integral $\tilde{A}^{CM}$, Eq.(14) covers all cases in which the conformal invariance of the Lagrangian density is broken by constant σ_0^2 and possible non–zero VEVs of other scalar fields that may be involved in the theory. Thence, we reach the following remarkable result:

Equivalence theorem of Schwinger (1969) [34]: *For $n = 4$, and only for $n = 4$, the conformal–invariant action of a scalar–ghost field interacting with zero–mass physical fields on a Riemann manifold of metric $g_{\mu\nu}(x)$ is generally equivalent to a (non–conformal–invariant) Hilbert–Einstein gravitational action on a Cartan manifold of conformal fundamental tensor $\tilde{g}_{\mu\nu}(x) = e^{2\alpha(x)}g_{\mu\nu}(x)$, plus an integral–action component containing mass terms for physical scalar fields.*

3.3 Action integrals of gauge fields are conformal invariant only in 4D

Proof: The Lagrangian density of a covariant Yang–Mills field A^{YM}_μ in nD is

$$L_{YM} = -\frac{1}{4}\mathrm{Tr}[\boldsymbol{F}_{\mu\nu}\boldsymbol{F}^{\mu\nu}]\,,\ \text{with } \boldsymbol{F}_{\mu\nu} = \sum_a \boldsymbol{\tau}_a F^a_{\mu\nu}\,,\ F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^a_{bc} A^b_\mu A^c_\mu\,,$$

where $\boldsymbol{\tau}_a$ are group–generator matrices, f^a_{bc} group structure constants and g the interaction constant. The symmetric energy–momentum tensor and its trace are respectively

$$\Theta^{YM}_{\mu\nu} = -\mathrm{Tr}[\boldsymbol{F}_{\mu\lambda}\boldsymbol{F}^\lambda_\nu] + \frac{1}{4}g_{\mu\nu}\mathrm{Tr}[\boldsymbol{F}_{\rho\sigma}\boldsymbol{F}^{\rho\sigma}]\,,\quad \Theta^{YM} = \Big(\frac{n}{4} - 1\Big)\mathrm{Tr}[\boldsymbol{F}_{\mu\nu}\boldsymbol{F}^{\mu\nu}]\,,$$

showing that the conformal invariance holds only for $n = 4$.

3.4 Action integrals of spinor fields can be conformal invariant in any dimension

Proof: The Lagrangian density of a free fermion field ψ of mass M on the Riemann manifold has the form

$$L_F = \frac{i}{2}\Big[\bar{\psi}(\not{D}\psi) - (\not{D}\bar{\psi})\psi\Big] + M\bar{\psi}\psi\,, \text{ with } \not{D} = e_a^\mu \gamma^a \partial_\mu - i\Gamma_\mu\,,$$

where e_a^μ is the "n-bein" (vierbein in 4D), γ^a a are the Dirac matrices in nD, Γ_μ are the spin operators which are necessary to make partial derivatives ∂_μ covariant [38], and the bar over ψ represents the covariant Hermitian conjugation of ψ in nD. The energy–momentum trace is $\Theta^F = M\bar{\psi}\psi$, implying conformal invariance for $M = 0$.

Another remarkable property of 4D physics is the following:

3.5 Spherical bodies are stable only in 4D

This is ensured by Birkhoff's theorem [1], which states that radially oscillating spherical bodies or spherical matter aggregations cannot radiate in 4D, since both the electromagnetic field and the gravitational field lack zero–spin modes precisely in 4 spacetime dimensions [33].

In conclusion

- *Conformal invariance and* 4*-dimensionality of spacetime are closely related, since non–trivial conformal–invariant actions of matter and gravity exist only in* 4*D.*
- *Einstein's GR can be incorporated into Cartan's CGR, provided that conformal symmetry is spontaneously broken.*
- *Matter-field Lagrangian densities on the Riemann and Cartan manifold maintain the same algebraic expression, all quantities being replaced by tilde quantities, whereas the form of the geometric Lagrangian density changes considerably.*
- *In* 4*D, and only in* 4*D, the conformal symmetry of an action integral on a Riemann manifold containing a scalar ghost field* $\sigma(x) = \sigma_0 e^{\alpha(x)}$, $\sigma_0 > 0$, *breaks down spontaneously to a Hilbert–Einstein action integral on the Cartan manifold, with* $g(x)$ *playing the role of the determinant of the fundamental tensor. Dimensional*

constant σ_0, which works as the conformal symmetry-breaking parameter, is related to gravitational constant κ by the equation $\sigma_0^2 = 6/\kappa = 6M_{rP}^2$.

4 The conformal background of the universe

As already explained in subsec.2.2, the fundamental tensor on a 4D Cartan manifold can be written as $\tilde{g}_{\mu\nu}(x) = e^{2\alpha(x)} g_{\mu\nu}(x)$ and the parameters of the manifold can be chosen so as to fulfill the condition $\sqrt{-g(x)} = 1$ for the determinant $g(x)$ of matrix $[g_{\mu\nu}(x)]$. The physical meaning of this factorization was clarified by Gürsey in 1963 [19].

4.1 Mach–Einstein Principle and conformal geometry

According to the Mach–Einstein doctrine, often referred to as the *Mach Principle*, the basic inertial frame is defined by distant bodies (now to be understood as galaxy clusters), the existence of such a frame being ensured by the observed simplicity of the universe on a sufficiently large scale. Unfortunately, however, one cannot infer the Mach Principle from Einstein's equations or vice versa, since, in the theoretical framework of GR, the inertial frame of a body is well–defined also in the absence of surrounding bodies.

To overcome this difficulty, Gürsey hypothesized that the fundamental tensor of spacetime geometry has a part $c_{\mu\nu}(x)$ which describes a conformally flat and spatially homogeneous geometry, and another part which describes the deviations from this uniform structure. The frame of distant bodies may then be defined as one in which $c_{\mu\nu}(x)$ takes a conformal form, so that light in this system travels on a straight line with constant velocity $c\,(=1)$. The boundary conditions for the metric will then require the fundamental tensor to tend asymptotically to a conformal metric characteristic of a uniform cosmological structure in the inertial frame. In this way, a transformation which takes the observer from an inertial to a non–inertial frame may be interpreted as a transformation which distorts the uniform and isotropic aspect of the cosmological background.

Clearly, $\tilde{g}_{\mu\nu}(x)$ is the only sort of fundamental tensor consistent with this view. The metric tensor factor can then be written as

$$g_{\mu\nu}(x) = \eta_{\mu\nu} + \Delta g_{\mu\nu}(x)\,, \ \text{ with } \ \det[\eta_{\mu\nu}] = \det[\eta_{\mu\nu} + \Delta g_{\mu\nu}(x)] = -1\,,$$

where $\eta_{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)$ is the flat metric tensor and $\Delta g_{\mu\nu}(x)$ describes the local irregularities of the metric. The conformally flat background is then described by $c_{\mu\nu}(x) = e^{2\alpha(x)}\eta_{\mu\nu}$, and the information on the cosmological structure is contained in the determinant of $\tilde{g}_{\mu\nu}(x)$ and tensor $\Delta\tilde{g}_{\mu\nu}(x) = e^{2\alpha(x)}\Delta g_{\mu\nu}(x)$. However, for the needs of our investigation we can assume $\Delta g_{\mu\nu}(x)$ to be very small, so that the gravitational field is represented as a small perturbation of the flat tensor, in which case the unit–determinant condition implies $\eta^{\mu\nu}\Delta g_{\mu\nu}(x) = 0$. This means, in practice, that black holes are ignored or replaced by extended bodies of large mass density - not so large, however, as to produce appreciable non–linear gravitational effects.

Functions $\alpha(x)$ were assumed by Gürsey to obey boundary conditions

$$\lim_{x^0 \to 0^+} \alpha(x) = \alpha_0\,, \qquad \lim_{x^0 \to +\infty} \alpha(x) = \alpha_\infty\,, \tag{15}$$

where α_0, α_∞ are finite constants satisfying the inequality $\alpha_0 < \alpha_\infty$. These conditions are intended to reflect the assumption that spacetime is conformally flat, both at the moment of the spontaneous breaking event and in the far future. However, in fixing these limits there is a certain arbitrariness, because, under the Riemann–manifold coordinate rescaling $x^\mu \to e^{-\lambda}x^\mu$, the above constants undergo changes $\alpha_0 \to \alpha_0 - \lambda$, $\alpha_\infty \to \alpha_\infty - \lambda$. Most authors choose $\alpha_0 = 0$, but in our view it is preferable to choose $\lambda = \alpha_\infty$, so that $\sigma(x)$ converges to σ_0 in the infinite future while $\sigma(0) = \sigma_0 e^{-\alpha_\infty}$. Thus, as an important bonus, *the Riemann and the Cartan manifold representations tend to coincide for* $x_0 \to +\infty$.

As it will be manifest in the following, this choice allows us to investigate and interpret the dynamic effects on matter produced by the conformal background in a manner significantly different from that provided by Gürsey and other authors.

4.2 General form of conformal geodesic equations

On the Cartan manifold, the motion of a point–like test particle under the action of the conformal gravitational field is governed by the geodesic equation

$$\frac{d^2x^\lambda}{d\tilde{s}^2}+\tilde{\Gamma}^\lambda_{\mu\nu}\frac{dx^\mu}{d\tilde{s}}\frac{dx^\nu}{d\tilde{s}}=0\,,\quad\text{or}\quad \frac{dx^\rho}{d\tilde{s}}\Big(\tilde{D}_\rho\frac{dx^\lambda}{d\tilde{s}}\Big)=0\ \text{(self–parallelism condition)}\,,$$

where $d\tilde{s}=\sqrt{\tilde{g}_{\mu\nu}dx^\mu dx^\nu}$ is the proper–time element of the particle along the geodesic [19] [23], $\tilde{\Gamma}^\lambda_{\mu\nu}$ are the Christoffel symbols constructed out of $\tilde{g}_{\mu\nu}$ and $\tilde{D}_\rho$ the covariant derivatives on the Cartan manifold. From Eq.(6) of sec.(2.2), we derive

$$\tilde{\Gamma}^\lambda_{\mu\nu}=\Gamma^\lambda_{\mu\nu}+\sigma^{-1}\big(\delta^\lambda_\mu\partial_\nu\sigma+\delta^\lambda_\nu\partial_\mu\sigma-g_{\mu\nu}\partial^\lambda\sigma\big)\,,$$

where $\Gamma^\lambda_{\mu\nu}$ are the Christoffel symbols constructed out of $g_{\mu\nu}$. The geodesic equation on the Riemann manifold then writes

$$\frac{d}{ds}\Big(\sigma g_{\mu\nu}\frac{dx^\nu}{ds}\Big)=\partial_\mu\sigma+\frac{1}{2}\,\sigma\,\big(\partial_\mu g_{\nu\lambda}\big)\frac{dx^\nu}{ds}\frac{dx^\lambda}{ds}\,,\ \text{ where } ds=\frac{\sigma_0}{\sigma}\,d\tilde{s}\,,$$

from which we extract the covariant 4D–acceleration of the test particle

$$a_\mu\equiv g_{\mu\nu}\frac{d^2x^\nu}{ds^2}=\frac{1}{\sigma}\Big(\partial_\mu\sigma-\frac{d\sigma}{ds}g_{\mu\nu}\frac{dx^\nu}{ds}\Big)+\frac{1}{2}\big(\partial_\mu g_{\nu\lambda}\big)\frac{dx^\nu}{ds}\frac{dx^\lambda}{ds}\,.$$

Posing $\vec{a}=\big\{a^1,a^2,a^3\big\}$, $u^\mu=dx^\mu/ds$, $\vec{u}=\big\{u^1,u^2,u^3\big\}$, $\partial^i=-\partial_i$, where $i=1,2,3$, and $\vec{\nabla}=-\big\{\partial^1,\partial^2,\partial^3\big\}$, we obtain the contravariant 3D–acceleration vector

$$\vec{a}\equiv\frac{d^2\vec{x}}{ds^2}=-\vec{\nabla}\alpha-\frac{d\alpha}{ds}\vec{u}-\frac{1}{2}\big(\vec{\nabla}g_{\nu\lambda}\big)u^\mu u^\nu,$$

where $\alpha=\ln\sigma$, clearly showing that the dilation field also exerts a force.

4.3 Synchronized comoving observers

These results give us the opportunity to introduce a precise definition of "comoving observers".

As well–known in tensor analysis, a complete set of geodesics stemming from a point V on a Riemann manifold can be used to define a system of *polar geodesic coordinates* [27], which are the analog of polar coordinates in Euclidean space. Let

us take as time–like coordinate distance τ along each geodesic measured from V, and as space–like coordinates the hyperbolic–coordinate "angles" $\hat{x} = \{\rho, \vartheta, \varphi\}$ of the line-element near V (Fig.1), as in

$$ds^2 = d\tau^2 - \tau^2\big[d\rho^2 + (\sinh\rho)^2 d\vartheta^2 (\sinh\rho\sin\vartheta)^2 d\varphi^2\big]\,,$$

where ds is the line-element of the metric.

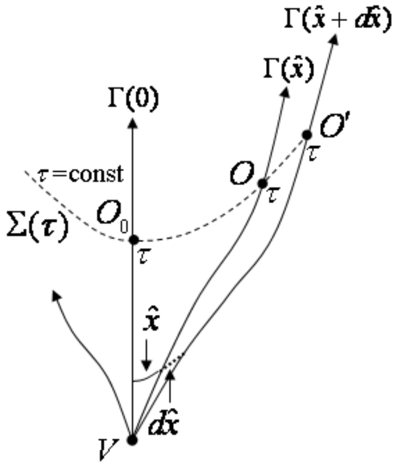

Fig.1. Polar geodesics stemming from a point V spanning a future cone in expansion and parameterized by hyperbolic coordinates $\{\tau, \hat{x}\}$. $\Sigma(\tau)$ is the locus of common time–like coordinate τ.

Since for each polar geodesic we have $d\tau/ds = 1$ and $d\hat{x}^i/ds = 0$, we also have $ds = d\tau$, $d\hat{x}^i = 0$, hence $\hat{x}^i =$ constant. Thus, $\hat{x}^i$ can be used to label the polar geodesics as $\Gamma(\hat{x})$. We can then cast the line–element in the form

$$ds^2(x) = d\tau^2 - \tau^2\, a_{ij}(\tau, \hat{x})\, d\hat{x}^i\, d\hat{x}^j\,,$$

for $i, j = 1, 2, 3$, with the conditions

$$\lim_{\tau\to 0} a_{11} = 1,\ \ \lim_{\tau\to 0} a_{22} = (\sinh\rho)^2,$$
$$\lim_{\tau\to 0} a_{33} = (\sinh\rho\,\sin\vartheta)^2\ \ \lim_{\tau\to 0} a_{ij} = 0\,,$$

where $i \neq j$.

If the metric is not so curved as to require multi–chart representation, all the information about the gravitational field is completely incorporated in functions a_{ij}.

The arc of geodesic OO' joining two infinitely close points O, O' of the same τ is $OO' = \tau\sqrt{a_{ij}d\hat{x}^i d\hat{x}^j}$ and, consequently, we have

$$\frac{ds}{d\tau} = \sqrt{1 - \tau^2\, a_{ij}(\tau, \hat{x})\frac{d\hat{x}^i}{d\tau}\frac{d\hat{x}^j}{d\tau}}\,.$$

The set of all points of the same geodesic time–like coordinate τ of all polar geodesics in the spacetime region defined by a future cone forms a 3D surface $\Sigma(\tau)$. A point O, running along one of these geodesics, is intended to represent an observer on the Riemann manifold having τ as proper coordinate. Since observers are called

"comoving" provided that they move along with universe expansion, we must assume the expansion factor $e^{\alpha(x)}$ to be the same all over $\Sigma(\tau)$, i.e., $\alpha(\tau)$ depends only on τ, at least as long as only gravitational forces are in play. Thus, $\Sigma(\tau)$ represents the *set of synchronized comoving observers* at proper time τ.

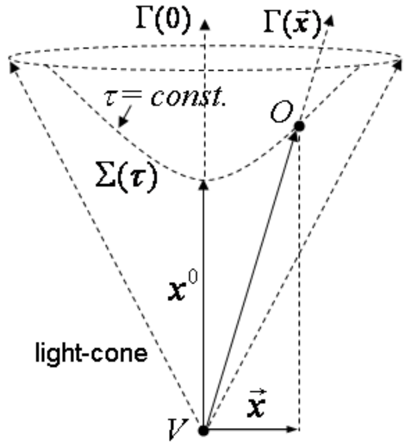

Fig.2. Polar geodesics spanning a future light cone in flat spacetime. Comoving observers lie on the 3D hyperboloid at kinematic–time distance $x^0 = \tau$ from cone–vertex V.

If the gravitational field is negligible, the future cone has the shape of the standard cone of special relativity (Fig.2), so that all geodesic stemming form V are straight lines. We can thus express τ as a function of Minkowski coordinates $\{x^0, \vec{x}\} \equiv \{x^0, x^1, x^2, x^3\}$ as $\tau = \sqrt{(x^0)^2 - |\vec{x}|^2}$, and therefore envisage $\vec{\nu}_O = \vec{x}/x^0$ as the velocity of comoving observer O at $\{x^0, \vec{x}\}$.

The following relationships are then easily proven

$$\frac{d\tau}{dx^0} \equiv \partial_0 \tau = \sqrt{1 - \nu_O^2}\,; \quad \vec{\nabla}\tau = -\frac{\vec{\nu}_O}{\sqrt{1 - \nu_O^2}}\,;$$

the first of which implies equation $x^0 = \tau$ on polar geodesic $\Gamma(0)$.

Passing from flat to general coordinates, we have $ds^2 = g_{\mu\nu} dx^\mu dx^\nu$ and may take as $\Gamma(0)$ the geodesic along which $x^i = 0$, $(i = 1, 2, 3)$, so that, on this geodesic, we also have $ds = \sqrt{g_{00}}\, dx^0$. Redefining x^0 so that $g_{00} = 1$, we still have the equation $ds/dx^0 = 1$ on $\Gamma(0)$ and therefore also $x^0 = \tau$. Consequently, the equations for $\partial_0\tau$ and $\vec{\nabla}\tau$ still hold, but now observer velocity $\vec{\nu}_O$ on the polar geodesic $\Gamma(\hat{x})$ at x depends in general on x.

Now let us consider a particle P running along a geodesic Γ_P possibly transverse to the set of $\Gamma(\hat{x})$. Let s_P be its proper time and x_P^μ their spacetime coordinates. Thus, ds_P is related to dx_P^μ by the equation $ds_P^2 = g_{\mu\nu}\, dx_P^\mu dx_P^\nu$ and its 4–velocity $u_P^\mu = dx_P^\mu/ds_P$ satisfy the equation

$$u_P^\mu u_{\mu\, P} \equiv g_{\mu\nu} \frac{dx_P^\mu}{ds} \frac{dx_P^\nu}{ds} = 1\,.$$

We can therefore write

$$u_P^0 = \frac{1}{\sqrt{1-\nu_P^2}}\,, \quad u_P^i = \frac{x_P^i}{ds_P} = \frac{x_P^i}{dx^0}\frac{x^0}{ds_P} = \frac{\nu_P^i}{\sqrt{1-\nu_P^2}}\,,$$

where $\vec{\nu}_P$ is the velocity of the particle at x. On account of the results obtained in the first part of this subsection, the 3D–acceleration of the particle writes

$$\vec{a}_P \equiv \frac{d^2\vec{x}_P}{ds_P^2} = \frac{\vec{\nu}_0 - \vec{\nu}_P}{\sqrt{1-\nu_0^2}}\frac{d\alpha}{d\tau} + \vec{a}_{GP}\,, \quad \vec{a}_{GP} = -\frac{1}{2}(\vec{\nabla} g_{\mu\nu})u_P^\mu u_P^\nu\,,$$

where $\vec{a}_{GP}$ is the gravitational acceleration of the particle. Here, the equations

$$\vec{\nabla}\tau = -\frac{\vec{\nu}_0}{\sqrt{1-\nu_0^2}}\,, \quad \frac{d\tau}{ds_P} = \frac{d\tau}{dx^0}\frac{dx^0}{ds_P} = \frac{\sqrt{1-\nu_P^2}}{\sqrt{1-\nu_0^2}}\,,$$

are used.

In conclusion

Whenever $d\alpha/d\tau > 0$, all particles are subjected to the viscous drag of the dilation field and consequently are forced to be at rest with respect to synchronous comoving observers. By contrast, whenever $d\alpha/d\tau < 0$, all particles move away from $\Sigma(\tau)$ with increasing velocity (but this, as we show in Part III, never happens).

4.4 Conformal uniformization of matter density during inflation

In the previous subsection, we assumed that scale–expansion factor $e^{\alpha(x)}$ is a pure function of τ, the proper time of comoving observers. The fact that all particles are forced by the viscous drag of the dilation field to be at rest with respect the comoving–observer space $\Sigma(\tau)$, which occurs only if $d\alpha/d\tau$ is positive and large enough, justifies the assumption only partly. The assumption would instead be perfectly correct if we were able to prove that all particles tend to be distributed uniformly on $\Sigma(\tau)$. In this subsection, we show that this, in fact, is the case, since, during scale expansion, massive particles become sources of dilation field, which makes all particles strongly repel each other. For our purposes, it is sufficient to prove that dilation field $\sigma_m(x)$ generated by a point–like particle of mass m at rest,

during scale expansion, acts repulsively on a test particle to an extent proportional to the mass of the latter.

As seen in subsec.(3.2), the action integral of CGR may be written in two equivalent ways: (1) on a 4D Riemann manifold R of metric tensor $g_{\mu\nu}(x)$, as a conformal invariant functional of the dilation field interacting with the gravitational field through the Ricci scalar tensor R; (2) on a 4D Cartan manifold C of fundamental tensor $\tilde{g}_{\mu\nu}(x)$, as a non–conformal invariant functional formally equal to the Hilbert–Einstein action. In the latter case, it has the general form $\tilde{A} = \tilde{A}^{CG} + \tilde{A}^{CM}$, with $\tilde{A}^{CG}$ and $\tilde{A}^{CM}$ respectively given by Eq.(13) with $\beta = 0$ and Eq.(14) of subsec.(3.2).

By variation of $\tilde{A}$ with respect to $\tilde{g}^{\mu\nu}(x)$, we obtain

$$\frac{\delta\tilde{A}^{CG}}{\delta\tilde{g}^{\mu\nu}(x)} = -\frac{\sqrt{-\tilde{g}}}{2\kappa}\Big[\tilde{R}_{\mu\nu}(x) - \frac{1}{2}g^{\mu\nu}(x)\tilde{R}(x) + \Lambda g^{\mu\nu}(x)\Big]\,, \quad \frac{\delta\tilde{A}^{CM}}{\delta\tilde{g}^{\mu\nu}(x)} = \frac{\sqrt{-\tilde{g}}}{2}\Theta^M_{\mu\nu}(x)\,,$$

from which we derive CGR equations

$$\tilde{R}_{\mu\nu} - \frac{\tilde{g}^{\mu\nu}}{2}\tilde{R} + \Lambda g^{\mu\nu} = \kappa\Theta^M_{\mu\nu} \;\text{ and }\; \tilde{R} = -\kappa\tilde{\Theta}^M + 4\Lambda\,,$$

where $\Lambda = -3\,\bar{c}\,\sigma_0^2/2 \simeq -10^{-52}$ m^2, as defined in Eq.(9), and $\kappa = 6/\sigma_0^2 \simeq 2.6 \times 10^{-70}$ m^2, in natural units, which are formally but not substantially identical to corresponding Hilbert–Einstein GR Eqs. (1) and (2) of subsec.(2.1). The difference, however, is revealed immediately by using Eqs. (7) and (8) of subsec.(2.2) for $n = 4$ and $\alpha(x) = \ln[\sigma(x)/\sigma_0]$, which yields

$$\begin{aligned}\tilde{R}_{\mu\nu} - \frac{\tilde{R}}{2}\,\tilde{g}_{\mu\nu} \;=\; & R_{\mu\nu} - \frac{R}{2}\,g_{\mu\nu} + \sigma^{-2}\big[4(\partial_\mu\sigma)(\partial_\nu\sigma) - g_{\mu\nu}g^{\rho\sigma}(\partial_\rho\sigma)(\partial_\sigma\sigma)\big] + \\ & 2\sigma^{-1}\big(g_{\mu\nu}D^2 - D_\mu\partial_\nu\big)\sigma = \frac{6}{\sigma_0^2}\tilde{\Theta}^M_{\mu\nu} + \frac{3\bar{c}\sigma_0^2}{2}\tilde{g}_{\mu\nu} = \frac{6}{\sigma^2}\Theta^M_{\mu\nu} + \frac{3\bar{c}\sigma^2}{2}g_{\mu\nu}.\end{aligned} \tag{16}$$

Note that $\Theta^M_{\mu\nu}(x)$ has dimension -2, as is evident from the variation with respect to $g^{\mu\nu}(x)$ of the conformal–invariant action integral on the Riemann manifold.

By contraction with $g^{\mu\nu} = e^{2\alpha}\tilde{g}^{\mu\nu}$ and using $\sigma(x)/\sigma_0 = e^{\alpha(x)}$, we obtain

$$-\,e^{2\alpha}\tilde{R} = -R + 6\,e^{-\alpha}D^2e^{\alpha} = \frac{6}{\sigma^2}\Theta^M + 6\bar{c}\,\sigma^2\,, \;\text{ or }\; D^2\sigma = \frac{\Theta^M}{\sigma} + \frac{R}{6}\,\sigma + \bar{c}\,\sigma^3\,, \tag{17}$$

with $\Theta^M = e^{-4\alpha}\tilde{\Theta}^M$, which can be regarded as the *motion equation of the dilation field* with ratio Θ^M/σ as dilation–field source. Using this equation, we can rewrite

Eq.(16) as

$$\begin{aligned} R_{\mu\nu} &= \frac{6}{\sigma^2}\Big(\Theta^M_{\mu\nu} - \frac{g_{\mu\nu}}{2}\Theta^M\Big) - g_{\mu\nu}\frac{3}{2}\,\bar{c}\,\sigma^2 + \sigma^{-1}\big(g_{\mu\nu}D^2\sigma + 2\,D_\mu\partial_\nu\sigma\big) - \\ &\quad \sigma^{-2}\Big[4(\partial_\mu\sigma)(\partial_\nu\sigma) - g_{\mu\nu}g^{\rho\sigma}(\partial_\rho\sigma)(\partial_\sigma\sigma)\Big]\,, \end{aligned}$$

showing that the source of the gravitational field includes the energy–momentum tensor of the dilation field. Note that, if the expansion factor $e^{\alpha(x)}$ is a pure function of τ, Eq.(17) implies that Θ^M and R must also share the same property. In any case, Eqs. (16), (17) range from

$$\begin{aligned} &\sigma^{-2}\Big[4(\partial_\mu\sigma)(\partial_\nu\sigma) - g_{\mu\nu}g^{\rho\sigma}(\partial_\rho\sigma)(\partial_\sigma\sigma)\Big] - \sigma^{-1}\,\big(g_{\mu\nu}D^2\sigma + 2\,D_\mu\partial_\nu\sigma\big) + \frac{3}{2}\,\bar{c}\,g_{\mu\nu}\,\sigma^2 = \\ &\qquad\qquad 6\,\sigma^{-2}\Big(\Theta^M_{\mu\nu} - \frac{g_{\mu\nu}}{2}\Theta^M\Big)\,, \\ &D^2\sigma \;=\; \Theta^M/\sigma\,, \end{aligned}$$

when the conformally flat condition $R = 0$ is fulfilled, to the familiar Einstein equations

$$R_{\mu\nu} = \kappa\Big(\Theta^M_{\mu\nu} - \frac{g_{\mu\nu}}{2}\Theta^M\Big)\,, \quad R = -\kappa\Theta^M - 4\Lambda\,,$$

which hold on the Riemann manifold when $\sigma(x)$ converges to σ_0 for $x^0 \to +\infty$.

Now, let us consider a point–like particle of mass m propagating along a world–line of the Cartan manifold described by the equation $x^\mu = z^\mu(\tilde{s})$, where $\tilde{s}$ is the proper time of the particle. As well–known from standard relativistic mechanics [23], here transferred to the Cartan manifold representation, the energy–momentum tensor and its trace have, respectively, the expressions

$$\tilde{\Theta}^m_{\mu\nu}(x) = m\,\tilde{\delta}^3\big(\vec{x} - \vec{z}(\tilde{s})\big)\tilde{u}_\mu(\tilde{s})\,\tilde{u}_\nu(\tilde{s})\frac{d\tilde{s}}{dx^0}\,, \quad \Theta^m(x) = m\,\tilde{\delta}^3\big(\vec{x} - \vec{z}(\tilde{s})\big)\frac{d\tilde{s}}{dx^0}\,,$$

where $\tilde{\delta}^3(\vec{x})$ is the 3D Dirac delta on the space–like sections of the Cartan manifold and $\tilde{u}_\mu(\tilde{s})$ is the covariant 4–velocity of the particle at its proper time $\tilde{s}$.

To pass from the Cartan to the Riemann manifold, we must use the equalities

$$\tilde{\delta}^3(\vec{x}) = e^{-3\alpha(x)}\delta^3(\vec{x})\,, \quad \tilde{u}_\mu(\tilde{s}) = e^{-\alpha(x)}u_\mu(s)\,, \quad \frac{d\tilde{s}}{dx^0} = e^{\alpha(x)}\frac{ds}{dx^0}\,,$$

where s is the proper time of the particle on the Riemann manifold which corresponds to $\tilde{s}$. These equations can easily be inferred, as the Dirac delta has dimension -3,

and $u^\mu = dx^\mu/ds$ and $u_\mu = g_{\mu\nu}u^\mu$. Moreover, since the spontaneous breaking of the conformal symmetry requires that the mass be written as $m = \gamma_m\sigma_0$, where γ_m is a positive adimensional constant, passing from the Cartan to the Riemann manifold we must perform the inverse Weyl transformation

$$m \equiv \gamma_m\sigma_0 \to \gamma_m\sigma(x) \equiv m_c(x)\,.$$

Function $m_c(x)$ is called the conformal mass of the particle. For $m = 1$ GeV/c^2, we have $\gamma_m = 1.68 \times 10^{-19}$. Hence, the correct relations between the corresponding tensors on the two manifolds are

$$\begin{aligned} e^{2\alpha(x)}\tilde{\Theta}^m_{\mu\nu}(x) &= \Theta^m_{\mu\nu}(x) = \gamma_m\,\sigma(x)\,\delta^3\big(\vec{x}-\vec{z}(s)\big)\,u_\mu(s)\,u_\nu(s)\frac{ds}{dx^0}\,,\\ e^{4\alpha(x)}\tilde{\Theta}^m(x) &= \Theta^m(x) = \gamma_m\,\sigma(x)\,\delta^3\big(\vec{x}-\vec{z}(s)\big)\frac{ds}{dx^0}\,. \end{aligned}$$

If the particle is at rest at a point x of polar geodesic $\Gamma(0)$, the tensors take the simple forms

$$\begin{aligned} \tilde{\Theta}^m(x) &= \tilde{\Theta}^m_{00}(x) = m\,\tilde{\delta}^3(\vec{x})\,, \quad \tilde{\Theta}^m_{0i}(x) = \tilde{\Theta}^m_{ij}(x) = 0\ (i \neq j)\,;\\ \Theta^m(x) &= \Theta^m_{00}(x) = \gamma_m\sigma(x)\,\delta^3(\vec{x})\,, \quad \Theta^m_{0i}(x) = \Theta^m_{ij}(x) = 0\ (i \neq j)\,. \end{aligned}$$

Consistent with these equations, for a continuous distribution of particles uniformly distributed on comoving–observer space $\Sigma(\tau)$ of the Riemann manifold, we have an energy–momentum tensor trace of the form

$$\Theta^B(x) = \gamma_B(\tau)\sigma(x)\,.$$

Now let us assume that the total energy–momentum tensor on the Riemann manifold splits into a background part Θ^B and a point–like particle part Θ^m. Since in these circumstances spacetime curvature R around the particle is zero everywhere, the motion equation of the dilation field expands to

$$D^2\sigma(x) = \frac{\Theta^B(x) + \Theta^m(x)}{\sigma(x)} + \bar{c}\,\sigma^3(x) = \gamma_B(\tau) + \gamma_m\delta^3(\vec{x}) + \bar{c}\,\sigma^3(x)\,.$$

Now let us pose $\sigma(x) = \sigma_B(\tau) + \sigma_m(x)$, where $\sigma_B(\tau)$ is the solution to the equation

$$D^2\sigma_B(\tau) = \frac{\Theta^B(x)}{\sigma(x)} + \bar{c}\,\sigma_B^3(\tau) = \gamma_B(\tau) + \bar{c}\,\sigma_B^3(\tau)\,.$$

We then obtain for $\sigma_m(x)$ the equation

$$D^2\sigma_m(\tau,\vec{x}) = \gamma_m \delta^3_a(\vec{x}) + \bar{c}\big[3\,\sigma_B^2(\tau)\,\sigma_m(\tau,\vec{x}) + 3\sigma_B(\tau)\,\sigma_m^2(\tau,\vec{x}) + \sigma_B^3(\tau)\big]\,. \qquad (18)$$

Here $\delta^3_a(\vec{x})$ is a 3D Gaussian of semi–width a, which replaces the Dirac delta in order to have some control over the solution singularity at $\vec{x}=0$. This is indeed possible, as $\bar{c} \simeq 3.65 \times 10^{-122}$ and the coefficients of the powers of $\sigma_m(x)$ within the square brackets:

$$3\,\bar{c}\,\sigma_B^2(\tau) \le 3\,\bar{c}\,\sigma_0^2 \simeq 2.0\times 10^{-52}\mathrm{m}^{-2}\,, \quad 3\,\bar{c}\,\sigma_B(\tau) \le 3\,\bar{c}\,\sigma_0 \simeq 4.7\times 10^{-87}\mathrm{m}^{-1}\,,$$

where the last inequality of Eqs. (15) of subsec.(4.1) is taken into account, are very small. Of note, an equation similar to (18) was first found by Gürsey [19] through heuristic considerations. On the Riemann manifold, as long as the metric is conformally flat, we have $R=0$. Hence, neglecting the second and third square–bracket terms of Eq.(18), we obtain the approximate equation

$$\Delta\sigma_m(\tau,\vec{x}) = -\gamma_m\delta^3_a(\vec{x}) - 3\,\bar{c}\,\sigma_B^2\sigma_m(\tau,\vec{x})\,,$$

where Δ is the Poisson operator $\partial_1^2+\partial_2^2+\partial_3^2$, whose solution for $r\equiv|\vec{x}|\gg a$ writes

$$\sigma_m = \frac{\gamma_m}{4\pi r}\,e^{-r/r_B(\tau)},$$

where $r_B(\tau) = 1/\sqrt{3\,\bar{c}}\,\sigma_B(\tau) < 7\times 10^{25}\,\mathrm{m} \simeq 2.27\times 10^3\,\mathrm{Mpc} \simeq 500$ galaxy–cluster diameters, is the range of the dilation field generated by the particle. It is then easy to prove that the approximation of the Dirac delta works well for any value of a. Instead, in order for this potential to be negligible with respect to $\sigma_B(\tau)$ for all values of r, the following inequality must hold

$$\frac{\gamma_m}{4\pi a} \ll \sigma_B(\tau) \le \sigma_0\,, \quad \text{i.e., } a \gg \frac{\gamma_m}{4\pi\sigma_0} \simeq 3.13\times 10^{-55}\,\mathrm{m}\,,$$

which can certainly be satisfied by a value of a enormously smaller than the Planck length $l_P \simeq 1.62\times 10^{-35}\,\mathrm{m}$. We can then safely write the 3D–acceleration of a test particle on the Riemann manifold as

$$\vec{a}_P = -\vec{\nabla}\alpha \simeq -\frac{1}{\sigma_B}\vec{\nabla}\big(\sigma_B+\sigma_m\big) = \frac{\vec{\nu}_O}{\sqrt{1-\nu_O^2}}\frac{d\alpha}{d\tau} + \frac{m}{4\pi\sigma_B^2}\Big(\frac{1}{r^2}+\frac{1}{r_B r}\Big)\frac{\vec{r}}{r}e^{-r/rB}\,,$$

whose term depending on m clearly exhibits the character of a central repulsive force.

Note that the repulsive force is very strong during the initial period of cosmic inflation, since, as shown in Part III, σ_B is initially very small. Apparently, for $\tau \to +\infty$ this term converges to

$$\vec{a}_m = \kappa \frac{m}{24\pi}\Big(\frac{1}{r^2} + \frac{1}{r_B r}\Big)\frac{\vec{r}}{r} e^{-r/rB} \, ,$$

but this is illusory, since at this limit $-R\sigma/6$ exactly cancels Θ^M in the motion equation of the dilation field and, correspondingly, the dilation field is constant.

By contrast, for $x^0 \to +\infty$, $\sigma(x)$ converges to σ_0 and R converges to $-\kappa\, \Theta^M - 4\,\Lambda$. Consequently, the gravitational equation decomposes into the equations

$$\begin{aligned} &R_{00} = \frac{3}{\sigma_0^2}\, \Theta^m_{00} + 2\,\Lambda = \frac{\kappa}{2}\, m\, \delta^3(\vec{x}) + 2\,\Lambda\, ; \quad R_{0i} = 0\, ; \\ &R = R_{00} - \sum_i R_{ii} = -\kappa\, m\, \delta^3(\vec{x}) - 4\,\Lambda\, ; \end{aligned}$$

while $R_{ij} = 0$ $(i \neq j)$ are determined by the requirement of spherical symmetry and condition $\sqrt{-g} = 1$. These equations can be solved in the linear approximation by posing

$$g_{\mu\nu}(x) = \eta_{\mu\nu} + h_{\mu\nu}(x)\, , \quad h_{00} = \varphi_m(x)\, , \quad h_{0i} = h^\mu_\mu = 0\, , \quad h_{ij} = 0 \ (i \neq j)\, ,$$

where $\varphi_m(x)$ is the static Newtonian potential of the particle and $h_{\mu\nu}$ are dealt with as infinitesimal quantities. Since in the assumed approximation the $\Gamma \times \Gamma$ products of the Riemann tensor are negligible [23] and $g_{\mu\nu}$ does not depend on x^0, we obtain the simple expressions

$$\begin{aligned} &R_{00} \simeq \partial_\mu \Gamma^\mu_{00}\, ; \quad R_{0i} \simeq \partial_j \Gamma^j_{0i} = 0\, ; \quad R_{ij} \simeq \partial_k \Gamma^k_{ij} - \partial_i \Gamma^\lambda_{j\lambda} \ (i \neq j)\, ; \\ &R_{00} \simeq \frac{1}{2}\Delta g_{00} = \frac{1}{2}\Delta\varphi_m\, ; \quad R_{0i} \simeq 0\, ; \quad R = R_{00} - \sum_i R_{ii} = \frac{1}{2}\Delta\varphi_m\, ; \end{aligned}$$

the first of which gives equation

$$\Delta\varphi_m = -\kappa\, \Theta^m - \Lambda r^2 = -\kappa\, m\, \delta^3(\vec{x}) - 4\,\Lambda\, ,$$

whose solution is

$$\varphi_m = \kappa \frac{m}{4\pi r} - \Lambda r^2 \,.$$

Here, the first term on the left represents the Newtonian potential of the particle and the second term a repulsive potential. Since $\Lambda \simeq -10^{-52}$ m^{-2}, the latter is very small, but sufficient to explain the observed acceleration of universe expansion. Hence we have

$$h_{00} = \kappa \frac{m}{4\pi r} - \Lambda r^2 \,; \quad h_{0i} = 0\,; \quad h_{ij} = \Big(\kappa \frac{m}{4\pi r^3} - \Lambda\Big)\, x_i x_j \ (\text{so that } h^\mu_\mu = 0)\,.$$

Now, solving the geodesic equation of the test particle we obtain

$$\vec{a}_P = -\frac{\kappa\, m}{4\,\pi\, r^2}\frac{\vec{r}}{r} - 2\,\Lambda \vec{r}\,,$$

which is the centripetal acceleration in the Newtonian approximation imparted by the gravitational force on a body of mass m, plus a small repulsive term radial to the comoving observers' space. Note that this small repulsive term may be the relic of a not yet exhausted conformal–background expansion.

In conclusion

During the scale expansion of spacetime, all particles are forced to be at rest and strongly repel each other all over the 3D space of synchronized comoving observers. This provides a very simple explanation of why the universe appears homogeneous and isotropic on the large scale. After expansion, all particles follow the laws of GR.

5 General conditions for matter production

Let us recall the geometry action–integral represented by Eq.(12) in subsec.(3.2). If the fundamental tensor of the Cartan manifold is conformally flat, which is presumed to be the case during the initial stage of universe expansion, both the Weyl–curvature tensor and the scalar Ricci tensor vanish, and therefore the geometry action–integral on the Riemann manifold and the motion equation for σ become, respectively,

$$A^G = -\int \frac{1}{2}\Big[\eta^{\mu\nu}(\partial_\mu\sigma)(\partial_\nu\sigma) - 2\,U(\sigma^2, \mathbf{\Psi})\Big] d^4x\,; \quad \partial^2\sigma + \frac{\delta U(\sigma^2, \mathbf{\Psi})}{\delta\sigma} = 0\,; \qquad (19)$$

where $\partial^2 = \eta^{\mu\nu}\partial_\mu\partial_\nu$.

As already remarked in subsec.(3.2), U must depend quadratically on σ otherwise this field may change sign, in contrast with equation $\sigma = \sigma_0\, e^\alpha$ with $\sigma_0 > 0$. However, quadratic couplings with zero–mass vector fields, as in $\sigma^2 g^{\mu\nu} A^a_\mu A^b_\nu$, must be excluded, since gauge–vector fields cannot incorporate ghosts as longitudinal spin components via the Englert–Brout–Higgs mechanism [13] [22]. Couplings with massless fermion fields are also excluded, as in conformal–invariant expressions scalar fields can couple only linearly with fermions. Thus, $\boldsymbol{\Psi}$ comprises at most only zero–mass scalar fields. In consideration of this, let us hereafter put $\boldsymbol{\Psi} = \boldsymbol{\varphi} = \{\varphi_1, \varphi_2, \dots \varphi_M\}$, where $M \geq 1$ and φ_k are real physical scalar fields.

From the first of Eqs. (19) we obtain energy–momentum tensor

$$\Theta^G_{\mu\nu} = -(\partial_\mu\sigma)(\partial_\nu\sigma) + \frac{\eta_{\mu\nu}}{2}\eta^{\rho\sigma}(\partial_\rho\sigma)(\partial_\sigma\sigma) - \frac{1}{6}\big(\eta_{\mu\nu}\partial^2 - \partial_\mu\partial_\nu\big)\sigma - \eta_{\mu\nu}U(\sigma^2, \boldsymbol{\varphi})\,.$$

Its trace is obtained by expanding the partial derivatives and using the motion equation, which yields

$$\Theta^G = \sigma\,\frac{\delta U(\sigma^2, \boldsymbol{\varphi})}{\delta\sigma} - 4\,U(\sigma^2, \boldsymbol{\varphi})\,.$$

This means that A^G is conformal invariant only if $U(\sigma^2, \boldsymbol{\varphi})$ is a monomial of fourth order in σ, i.e., U does not depend on $\boldsymbol{\varphi}$. As for the continuity equation of $\Theta^G_{\mu\nu}$, we obtain

$$\partial^\mu\Theta^G_{\mu\nu} = \frac{\delta U(\sigma^2, \boldsymbol{\varphi})}{\delta\sigma}\,\partial_\nu\sigma - \partial_\nu U(\sigma^2, \boldsymbol{\varphi}) = -\frac{\delta U(\sigma^2, \boldsymbol{\varphi})}{\delta\varphi_k}\,\partial_\nu\varphi_k\,, \tag{20}$$

where the motion equation for σ was again used and summation over φ subscripts is understood.

Here we see that, unlike the GR case, provided that $U(\sigma^2, \boldsymbol{\varphi})$ is a function of both σ and $\boldsymbol{\varphi}$, the energy–momentum current of the geometry is not conserved. Since the total energy–momentum current must be conserved, for compensation the energy–momentum tensor of matter must obey the continuity equation

$$\partial^\mu\Theta^M_{\mu\nu} = \frac{\delta U(\sigma^2, \boldsymbol{\varphi})}{\delta\varphi_k}\,\partial_\nu\varphi_k\,. \tag{21}$$

Clearly, Eqs. (20) and (21) exhibit an energy-momentum trade-off between geometry and matter. It is also clear that, in order for this trade-off to be really capable of

increasing the energy of matter, $U(\sigma^2, \boldsymbol{\varphi})$ must always increase for increasing values of σ and $|\varphi_k|$. Hence, on the Riemann and the Cartan manifold it has the general forms, respectively,

$$U(\sigma^2, \boldsymbol{\varphi}) = a\,\sigma^2 C^{hk}\varphi_h\varphi_k + b\sigma^4 + c\,, \quad \tilde{U}(\sigma_0^2, \tilde{\boldsymbol{\varphi}}) = a\,\sigma_0^2\, C^{hk}\tilde{\varphi}_h\tilde{\varphi}_k + b\,\sigma_0^4 + c\,,$$

where $a > 0$ and the matrix of coefficients C^{hk} is real symmetric and positive definite. As for b and c, the former must be negative in order for cosmological constant $\Lambda = b\,\sigma_0^4$ to be negative, and the latter is an arbitrary real constant.

Here we discover an important point of the theory. By a suitable rotation of field–multiplet $\boldsymbol{\varphi}$ we may bring U and $\tilde{U}$, respectively, to the forms

$$U(\sigma^2, \boldsymbol{\varphi}) = \sum_k c_k^2\,\sigma^2\,\varphi_k^2 + b\,\sigma^4 + c\,, \quad \tilde{U}(\sigma_0^2, \tilde{\boldsymbol{\varphi}}) = \sum_k c_k^2\,\sigma_0^2\,\tilde{\varphi}_k^2 + b\,\sigma^4 + c\,,$$

As an ingredient of the geometry Lagrangian density on the Cartan manifold, the term $-\sum_k a^2\sigma_0^2\tilde{\varphi}_k^2$ on the left hand side of the second equation can be moved to the matter Lagrangian density $\tilde{L}^M$ with the opposite sign, where, because of the negative sign, it imparts to $\tilde{\varphi}_k$ the character of tachyon fields rather than of standard scalar fields of particles of mass $a_k\,\sigma_0$. If positive potential–energy self–interaction terms of order 4 in $\tilde{\varphi}_k$ are also present, the part of matter Lagrangian density containing these fields becomes that of *a set of Higgs scalar fields*.

As shown in subsec.5.1 of Part II, the importance of this fact is closely related to the occurrence of a negative pressure term in the energy–momentum tensor of the matter field. To clarify this point, let us consider let us the GR equation on the Riemann manifold in its standard form

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda g_{\mu\nu} = \kappa\,\Theta_{\mu\nu}\,,$$

where $\Theta_{\mu\nu}$ is the energy–momentum tensor of a set of fields possibly including the dilation field. Thus, we have $D^\mu\Theta_{\mu\nu} = 0$, since the covariant divergence of the left member of the equation also is identically zero because of the second Bianchi identities.

Let us assume that the fields on which $\Theta_{\mu\nu}$ depends behave comprehensively as a fluid, with volume elements moving along the geodesics at covariant 4–velocity

$u_\mu = g_{\mu\nu} u^\mu$, as is indeed the case for matter fields under the action of rapid scale expansion, for the reasons discussed in the previous subsections. Hence, $\Theta_{\mu\nu}$ takes the simple form

$$\Theta_{\mu\nu} = (\rho + p)\, u_\mu u_\nu - g_{\mu\nu} p\,,$$

where ρ is the energy density of the fluid and p its local pressure. From the conservation equation $D_\mu\big[(\rho+p)\, u_\mu u_\nu - \delta^\nu_\mu p\big] = 0$ we derive the equation $\partial_\mu\big(\sqrt{-g}\,\rho\, u^\mu\big) = -p\,\partial_\mu(\sqrt{-g}\, u^\mu\big)$, where equation $u_\mu u^\mu = 1$ and geodesic equation $u^\nu D_\nu\, u_\mu = 0$, i.e., the self–parallelism condition, were used. In terms of synchronous comoving coordinates, we have $\partial_0 = \partial_\tau$, $u_0 = 1$, $\partial_i = 0$ $(i = 1, 2, 3)$. Thus, we obtain

$$\partial_\tau\big(\sqrt{-g}\rho\big) = -p\,\big(\partial_\tau\sqrt{-g}\big)\,,$$

which, when multiplied by volume element $d^3\hat{x}$, becomes $\delta M(\tau) = -p\,\delta V(\tau)$, which is the mass in volume element $\delta V(\tau) = \sqrt{-g}\, d^3\hat{x}$. Thus, independently of any other consideration, matter is created, provided that p is negative [2] [17].

In conclusion

The expansion of the universe fed by a trade–off of energy from geometry to matter is possible only in the general framework of a spontaneous breakdown of conformal symmetry, with an action integral grounded on a 4D Riemann manifold. In order for this to occur, two conditions must be fulfilled: 1) the matter Lagrangian density on the Riemann manifold must include one or more (massless) scalar fields quadratically coupled with the dilation field in such a way that the matter scalar fields on the Cartan manifold are of Higgs type; 2) the matter–field pressure must be negative during the initial stage of universe expansion.

6 On gravitational–field renormalizability

In the post–inflation era, the gravitational action integral on the Cartan manifold $\tilde{A}^G$ described by Eq.(13) of subsec.(3.2) coincides with that on the Riemann manifold:

$$A^{RG} = -\int \frac{\sqrt{-g(x)}}{2}\Big[\frac{\beta^2}{2}\, C_{\mu\nu\rho\sigma}(x) C^{\mu\nu\rho\sigma}(x) + M_{rP}^2\, R(x)\Big] d^4x\,, \qquad (22)$$

where the cosmological constant is set at zero for simplicity.

Unfortunately, the term proportional to $C^2 = C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma}$ contains a time derivative of order four, which makes the action integral equivalent to the sum of the action integral for standard zero–mass gravitons and an action integral for massive–graviton ghosts [28], i.e., massive gravitons with negative propagators. In the following, for the sake of conciseness, quantum superpositions of standard zero–mass gravitons and massive graviton ghosts are called *conformal gravitons*.

The purpose of this Section is to explain briefly why it is precisely this feature which makes quantum gravity renormalizable and spacetime locally flat, and also to propose an honorable solution to the problem of conformal graviton ghosts.

The identity

$$C^2 = R_{GB} + 2\Big(R_{\mu\nu}R^{\mu\nu} - \frac{1}{3}\,R^2\Big)$$

where $R_{GB} = R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 4R_{\mu\nu}R^{\mu\nu} + R^2$ is the Gauss–Bonnet term, can be proven. Since in 4D spacetime R_{GB} is a topological invariant, the variation of $\int\sqrt{-g(x)}\,R_{GB}(x)\,dx^4$ with respect to $g^{\mu\nu}(x)$ vanishes in any region topologically equivalent to flat spacetime. In other words, in any spacetime region free of singularities, Eq.(22) is equivalent to

$$A^{RG} = -\int\frac{\sqrt{-g(x)}}{2}\Big[\beta^2\Big(R_{\mu\nu}R^{\mu\nu} - \frac{1}{3}\,R^2\Big) + M_{rP}^2\,R(x)\Big]d^4x\,.$$

The proof of renormalizability of the quantum gravity equations derived from this action integral was carried out by K.S. Stelle in 1976 [35]. Proof of the asymptotic freedom of conformal gravitons was provided by Tomboulis in 1980 [36] and study of their interaction with matter by Antoniadis and Tomboulis in 1986 [37]. Here, we limit ourselves to reporting from Stelle [35] a few salient results, with some simplification and changes in annotation.

6.1 Conformal graviton propagators

The most convenient definition of the gravitational field variable is given in terms of contravariant metric density

$$h^{\mu\nu}(x) = \frac{1}{M_{rP}}\left[\sqrt{-g(x)}\,g^{\mu\nu}(x) - \eta^{\mu\nu}\right], \tag{23}$$

where $\eta^{\mu\nu} = \mathrm{diag}(1,-1,-1,-1)$, together with the *harmonic gauge* condition $\partial_\nu h^{\mu\nu} = 0$, which is satisfied provided that spacetime coordinates are harmonic, i.e., they satisfy the covariant D'Alembert equation $D_\nu D^\nu x^\rho = 0$.

In the linear approximation and with the exclusion of all gauge–dependent terms, the propagator of the conformal graviton in momentum space p_μ is

$$D_{\mu\nu\rho\sigma}(p) = \frac{1}{(2\pi)^4}\left[\frac{P^{(2)}_{\mu\nu\rho\sigma}(p)}{p^2(1-p^2/M_G^2)+i\epsilon} - \frac{2P^{(0)}_{\mu\nu\rho\sigma}(p)}{p^2+i\epsilon}\right], \tag{24}$$

where $M_G = M_{rP}/\beta$ and $P^{(2)}_{\mu\nu\rho\sigma}$, $P^{(0)}_{\mu\nu\rho\sigma}$ are respectively the rank–two polarization tensors of the spin–two and spin–zero components of the conformal graviton. These may be written as

$$P^{(2)}_{\mu\nu\rho\sigma} = \frac{1}{2}\big(P_{\mu\rho}P_{\nu\sigma} + P_{\mu\sigma}P_{\nu\rho}\big) - \frac{1}{3}P_{\mu\nu}P_{\rho\sigma}\,; \quad P^{(0)}_{\mu\nu\rho\sigma} = \frac{1}{3}P_{\mu\nu}P_{\rho\sigma}\,,$$

where $P_{\mu\nu} = \eta_{\mu\nu} - p_\mu p_\nu/p^2$ is the space–direction projector [35]. Eq.(24) can be rewritten as

$$D_{\mu\nu\rho\sigma}(p) = \frac{P^{(2)}_{\mu\nu\rho\sigma}(p)}{2\,\pi^4}\left[\frac{1}{p^2+i\epsilon} - \frac{1}{p^2-M_G^2+i\epsilon}\right] + \text{gauge terms}\,,$$

in which the decomposition of the conformal graviton propagator into a zero–mass graviton propagator and a massive graviton ghost is manifest. Note that, for $M_G < \infty$, the massive–ghost propagator is similar to the regulator introduced by Pauli and Villars in 1949 to ensure gauge–invariant renormalization of quantum electrodynamics [31]. Also note that for $M_G \to \infty$, the propagator tends to the familiar zero–mass graviton propagator

$$\hat{D}_{\mu\nu\rho\sigma}(p) = \frac{1}{(2\pi)^4}\frac{\hat{P}^{(2)}_{\mu\nu\rho\sigma}(p)}{p^2+i\epsilon} + \text{gauge terms}\,,$$

where $\hat{P}^{(2)}_{\mu\nu\rho\sigma} = P^{(2)}_{\mu\nu\rho\sigma} - 2P^{(0)}_{\mu\nu\rho\sigma}$ is the polarization tensor of the two elicity components. For $p^2 \gg M_G^2$, the gravitational field disappears, implying that, for $\beta \ll 1$, spacetime flattens at sub-Planckian distances from any point–like event. This also means that, in a sufficiently small neighborhood of each point, both Cartan and Riemann manifolds are conformally flat.

This is also evident in the static Newtonian limit, at which the static potential profiles of the standard and massive–ghost propagators in the coordinate space respectively have the forms

$$\hat{V}_0(r) \sim -\frac{1}{r}\,; \quad V_{M_G}(r) \sim \frac{e^{-M_G r}}{r}\,.$$

Thus, the conformal gravity potential $V(r) = V_0(r) + V_{M_G}(r)$ is virtually zero for $r \ll 1/M_G$, which shows that space is flat in the neighborhoods of all its points.

6.2 Living with ghosts

The occurrence of massive gravitational ghosts is almost universally considered an insurmountable difficulty of the theory - so much so that most theorists decided to look for alternatives to Einstein's theory of gravitation. The drawbacks caused by these ghosts ultimately regard the analytical properties of the S–matrix. However, there are three possible interpretations of this point, depending on how one decides to turn round the singularities of the S–matrix associated with ghost mass when unitarity equation $S^{\dagger}S = 1$ is evaluated [11]:

1. *Unitarity violation due to the presence of negative-norm states*

   This is the most popular interpretation, which is immediately suggested by the negative sign of ghost propagators.

2. *Lack of lower bound for gravitational energy*

   This is an alternative to point 1, since negative norms can be made positive by changing the sign of particle energy.

3. *Cause–effect inversion at distances $\leq 1/MG$ from point–like events*

   This interpretation was provided by Lee and Wick at the end of the 1960s [24] [25] as an alternative to interpretations 1 and 2, to support the Pauli–Villars' renormalization method [31]. As argued by these authors, the violation of causality on such small scales is virtually undetectable and may be ignored in practice. From a logical–theoretical standpoint, however, this sort of violation

is unacceptable, as it undermines the Principle of Causality, which is almost universally regarded as the most fundamental principle of physics (at least as far as the existence of superluminal neutrinos is not confirmed).

Let us accept interpretation 3, and imagine what sort of experiment should be performed in order to detect the violation of causality. The Large Hadron Collider of Geneva will never be able to focus enough energy on a target of radius comparable to the Planck length $l_P = 1.6 \times 10^{-35}$ m, but the fabulous rifle of Baron Münchausen certainly can (Fig.3):

Fig.3. Baron Münchausen trying to detect the violation of causality by shooting at a target of sub–Planckian radius $r_{sP} \equiv 1/(\beta M_{rP}) \ll l_P$, with $l_P \equiv 1/M_{rP} \simeq 1.62 \times 10^{-35}$m the Planck length.

But what would the Baron actually see through the prodigious telescope of his rifle? No matter how small the bullet, its effective size could not be smaller than radius Δr of its quantum–mechanical wave–packet.

The mean radius of the Gaussian wave–packet of a point–like probe particle is related to the mean radial impulsion Δp by the 3D Heisenberg indetermination relation $\Delta p\, \Delta r = 3\hbar/2$, where $\hbar$ is the Planck constant divided by $2\,\pi$. Hence, if Δr is of the order of magnitude of Planck length l_P, we can estimate that the packet has an effective mass of $M_r \simeq \Delta p/c = 3\hbar/l_P c$, with c as the speed of light, because the energy of the wave–packet is almost totally due to momentum indetermination. Instead, l_P is related to gravitational constant G by the equation $l_P = \sqrt{\hbar\, G/c^3}$ and the Schwartschield radius of a black hole of mass M_r is $r_S = 2\,G\,M_r/c^2 = 3\,G\,\hbar/l_P c^3$. This, combined with $G\,\hbar = l_P^2 c^3$, yields $r_S = 3\,l_P$: the radius of the black hole created at the impact point is three times larger than the wave–packet radius of the bullet! Thus, provided that $r_{sP} \ll l_P$, the Baron would simply observe that the violation of causality is censored by the event horizon of the small black hole created at the

point of impact.

In fact, this conclusion is conditioned by the validity of the censorship theorem of Hawking and Penrose [21] [32]. Roughly speaking, this theorem stipulates that, with the possible exception of the singularity that causes the BigBang, naked singularities of the gravitational field cannot exist, i.e., they are confined within black–hole horizons and are therefore unobservable.

In conclusion

CGR is both renormalizable and causality–preserving (with the possible exception of the conformal–symmetry breaking event) since in 4D gravitational ghosts of ultra–Planckian mass exist theoretically but not phenomenologically.

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